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Preface

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This special issue contains a collection of referred papers devoted to the memory of Sergio Campanato and also of Guido Stampacchia, one of his main teachers. Many of these papers are the text of the lectures delivered during the International Conference entitled "Variational Analysis and Partial Differential Equations" held at the International School of Mathematics "Guido Stampacchia" in Erice July 5–14, 2006 and devoted to the memory of Sergio Campanato and to the awarding of the 2nd Gold Medal Prize "Guido Stampacchia". In [18] a very extensive commemoration of Guido Stampacchia's life and work can be found; here it is only worth mentioning the outstanding impact of the variational and quasi-variational inequalities on the social and economic life: his early intuition of the extraordinary importance of this theory for solving wide classes of economic, financial and transportation equilibrium problems has been completely verified (see [21] and [17] for more information).

Sergio Campanato was one of the first pupils of Guido Stampacchia and also of Enrico Magenes and it is easy to see Guido Stampacchia's influence in the scientific education of Sergio Campanato, especially as concerns his interest in the regularization of the solutions to partial differential equations and systems.

As a matter of fact, the regularization theory was one of the predominant interests of Sergio Campanato. He had a peculiar idea regarding the way to achieve this objective, namely he was concerned with the way of finding a general regularization methodology from which the regularity results in all the concrete situations could be derived. He was able to develop this methodology which, roughly speaking, runs in the following way: one has to study how the

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ratio of the L^p norms on concentric balls of different radii of a solution of a class of second order differential systems depends on the ratio of the radii. For each class of systems (linear, quasilinear, nonlinear) and depending on the regularity assumptions of the coefficients, one obtains a different power of the ratio of the radii and, using appropriate Poincaré inequalities and the properties of the Campanato spaces $\mathcal{L}^{p,\lambda}$, one gets the Hölder continuity of solutions for low values of the dimension of the problem or, in general, the partial Hölder continuity of solutions.

It is worth mentioning how the above methodology works for second order nonlinear variational elliptic and parabolic systems. In these cases Sergio Campanato first shows differentiability results for the solutions of the so-called basic systems, namely systems of the type $\sum_{i=1}^{n} D^i a^i(Du) = 0$ or $-\sum_{i=1}^{n} a^i(Du) + \frac{\partial u}{\partial t} = 0$; then he proves some of Caccioppoli's type inequalities and the so-called fundamental estimates for the gradient of the solutions; subsequently, using elliptic or parabolic Poincaré inequalities and again Caccioppoli's type inequalities, he shows that the solutions belong to the Campanato space $\mathcal{L}^{p,\lambda}$, from which the Hölder continuity of the solutions to the basic systems follows for suitable values of the dimension of the problem. Finally, using the so-called "Campanato decomposition method", it is possible to derive similar estimates for the gradient of a solution to general nonlinear systems and to show that the solution belongs to the space $\mathcal{L}^{p,\lambda}$, from which the Hölder continuity of the solution follows.

This methodology runs not only for variational solutions, but also for nonvariational nonlinear systems of the parabolic and elliptic type, provided that a suitable ellipticity condition is satisfied.

Here we enter the second predominating field of interest of Sergio Campanato. In fact, starting from the 1960s, he was concerned with second order nonvariational partial differential equations with discontinuous coefficients and the related boundary value problems. For these kinds of problems the searched solution is of strong type, namely of class $W^{2,p}$ (or of class $W_p^{2,1}$ in the parabolic case) and in general, unlike for the variational problems, the usual ellipticity condition is not enough to ensure the well-posedness of the boundary value problems. Assuming that the so-called Cordes condition is fulfilled, Sergio Campanato was able to show the existence of a unique solution of the Dirichlet problem in the class $W^{2,p}(\Omega)$, $p \in (2-\varepsilon, 2+\varepsilon)$ and also some regularity results. In the 1990s, he was again concerned with the study of strong solutions for second order nonlinear nonvariational elliptic and parabolic systems with discontinuous coefficients. In order to solve this problem, he was led to introduce the theory of near operators, whose main results and definitions are the following (see [20] for details).

Suppose that $\mathcal{B}$ is a set, $\mathcal{B}_1$ is a real Banach space, and $A : \mathcal{B} \to \mathcal{B}_1$ is a mapping. Moreover, let B be another mapping such that $B : \mathcal{B} \to \mathcal{B}_1$.

Definition 1 We shall say that A is near B if there exist two positive constants α and K with $K \in (0, 1)$, such that $\forall u,\ v \in \mathcal{B}$, we have

$$\|B(u) - B(v) - \alpha\,[A(u) - A(v)]\,\|_{\mathcal{B}_1} \leq K\|B(u) - B(v)\|_{\mathcal{B}_1}.$$

Theorem 1 *The mapping $A : \mathcal{B} \to \mathcal{B}_1$ is injective, or surjective, or bijective if and only if it is near to a mapping $B : \mathcal{B} \to \mathcal{B}_1$ which is injective, or surjective, or bijective.*

A simple consequence of Theorem 1 is that if $B : \mathcal{B} \to \mathcal{B}_1$ is bijecive and if A is near to B, then, $\forall f \in \mathcal{B}_1$ there exists a unique $u \in \mathcal{B}$ such that $A(u) = f$ and $\|B(u) - B(v)\|_{\mathcal{B}_1} \leq \frac{\alpha}{1-K}\|f - A(v)\|_{\mathcal{B}_1} \forall v \in \mathcal{B}_1$.

Many other interesting consequences can be derived for near operators, but we would like to point out how the theory of near operators runs if $\mathcal{B}_1$ is a real Hilbert operator. We recall two new definitions.

Definition 2 A is said to be monotone with respect to B if $\forall u, v \in \mathcal{B}$ we have

$$(A(u) - A(v)|B(u) - B(v))_{\mathcal{B}_1} \geq 0.$$

A is said to be strictly monotone with respect to B if there exist two positive constants M and ν such that $\forall u, v \in \mathcal{B}$ we have

$$\|A(u) - A(v)\|_{\mathcal{B}_1} \leq M\|B(u) - B(v)\|_{\mathcal{B}_1}$$

$$(A(u) - A(v)|B(u) - B(v))_{\mathcal{B}_1} \geq \nu\|B(u) - B(v)\|^2_{\mathcal{B}_1}.$$

Then Sergio Campanato shows that "*Two mappings A and $B : \mathcal{B} \to \mathcal{B}_1$ are near if and only if A and B are strictly monotone*" and a generalized version of Lax-Milgram theorem easily follows assuming $\mathcal{B} = \mathcal{B}_1 = H$ a real Hilbert space and $B = I$ the identity function on H. Moreover the existence theory for variational nonlinear elliptic problems runs in a simple way in the case of strictly monotone operators.

Suppose now that A is a differential operator of the type $a(x, H(u))$ where x varies in a bounded open set $\Omega \subset \mathbb{R}^n$, of class C^2, $u : \Omega \to \mathbb{R}^N$ is a vector, N being an integer ≥ 1, $H(u) = \{D_{ij}u\}$ is an $n \times n$ matrix of vectors in $\mathbb{R}^N$, that is an element of $\mathbb{R}^{n^2N}$ and finally $a(x, \xi)$ is a vector of $\mathbb{R}^N$, measurable in x and continuous in ξ such that $a(x, 0) = 0$. Sergio Campanato gives a definition of ellipticity in the following way:

(A^2) *There exist three positive constants α, γ and δ such that $\gamma + \delta < 1$ and $\|\mathrm{Tr}\,\tau - \alpha\,(a(x, \xi + \tau) - a(x, \xi))\,\|^2_N \leq \gamma\|\tau\|^2_{n^2N} + \delta\|\mathrm{Tr}\,\tau\|^2_N$ for any $\xi, \tau \in \mathbb{R}^{n^2N}$ and almost all $x \in \Omega$ (note that $Tr\,\tau = \sum_{i=1}^{n} \tau_{ii}$ is an N-dimensional vector).*

Then Sergio Campanato shows the following theorem.

Theorem 2 *If the vector $a(x, \xi)$ is elliptic and the open set Ω is convex, then the operator $a(x, H(u))$ is near the operator Δu, both operators being understood as operators $H^2 \cap H^1_0(\Omega) \to L^2(\Omega)$.*

In order to understand the meaning of condition (A^2) the following characterization can be useful.

Theorem 3 *Let $a(x, \xi)$ satisfy condition (A^2). Then there exist three positive constants M, μ and K, $\mu > K$, such that for all $\tau, \xi \in \mathbb{R}^{n^2}$ and almost all $x \in \Omega$*

$$\|a(x, \xi + \tau) - a(x, \xi)\|_N \leq M\|\tau\|_{nN} \tag{1}$$

$$(a(x, \xi + \tau) - a(x, \xi)|\ Tr\,\tau)_N \leq \mu\|Tr\,\tau\|^2_N - K\|\tau\|^2_{n^2N} \tag{2}$$

where $\mu = \frac{1-\delta}{2\alpha}$, $K = \frac{\gamma}{2\alpha}$ and $M = \frac{\sqrt{\gamma}+\sqrt{nN}(1+\sqrt{\delta})}{\alpha}$. Vice versa if there exist three positive constants M, μ, K and $\mu > K$ such that (1) and (2) hold, then $a(x, \xi)$ satisfies (A^2) with constants $\alpha \in \left(0, \min \frac{1}{2\mu}, \frac{2(\mu-K)}{M^2}\right)$, $\delta = 1 - 2\alpha\mu$ and $\gamma = \alpha^2M^2 + 2\alpha K$.

Hence from Theorem 3 it follows that the meaning of the ellipticity condition of Sergio Campanato is that it requires a particular kind of strict monotonicity of the operators.

In particular from condition (1) of Theorem 3 it follows, by virtue of the Rademacher theorem, that partial derivatives $a^{ij}(x,\xi) = \frac{\partial a(x,\xi)}{\partial \xi_{ij}}$ exist for almost all $\xi \in \mathbb{R}^n N$ and we have

$$\sum_{i,j=1}^{n} \varsigma_i \varsigma_j \left(\frac{\partial a(x,\xi)}{\partial \xi_{ij}} \eta | \eta \right)_N \geq \frac{1-(\gamma-\delta)}{2\alpha} \|\varsigma\|_n^2 \|\tau\|_N^2$$

$\forall \varsigma \in \mathbb{R}^n$, $\forall \eta \in \mathbb{R}^N$ and for almost all $x \in \Omega$. Of course, for linear operators, (A^2) condition becomes the Cordes condition.

It is also worth mentioning that the theory of near mappings does not require any assumption on the set $\mathcal{B}$: it is not necessary that $\mathcal{B}$ is a linear subspace or a convex set and this fact allows us to study, by means of the nearness theory, also boundary problems with nonlinear boundary condition, e.g. the contact problems (see [19]).

The theory of near mappings works also for parabolic operators. In fact Sergio Campanato shows the following theorem:

Theorem 4 *If A, B, C are mappings $\mathcal{B} \to \mathcal{B}_1$ and if A is near B with constants α and K, C is monotone with respect to B, then $A + C$ is near $B + \alpha C$ with the same constants α and K and the problem*

$$\begin{cases} u \in \mathcal{B} \\ A(u) + C(u) = f \end{cases}$$

has a solution if and only if the problem

$$\begin{cases} u \in \mathcal{B} \\ B(u) + \alpha C(u) = f \end{cases}$$

has a solution.

From Theorem 4 we derive existence results for parabolic systems assuming $B(u) = \Delta u$, A near Δu, $C(u) = \frac{\partial u}{\partial t}$ and checking if $-\left(A(u-v) | \frac{\partial}{\partial t}(u-v)\right)_{\mathcal{B}_1} \geq 0 \; \forall u, v \in \mathcal{B}$.

We have omitted to speak about other fields of research of great interest which Sergio Campanato was concerned with and we conclude remarking that the contribution of Sergio Campanato to the regularity theory for nonlinear variational and nonvariational partial differential equations and systems, his outstanding and general regularization method, the theory of nearness for operators, his strong attempt to establish a unified theory from which it is possible to derive the solution of many of the most important and unsolved problems and his challenge for shining and profound mathematics represent an gift for all mathematicians and an improvement of our knowledge.

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